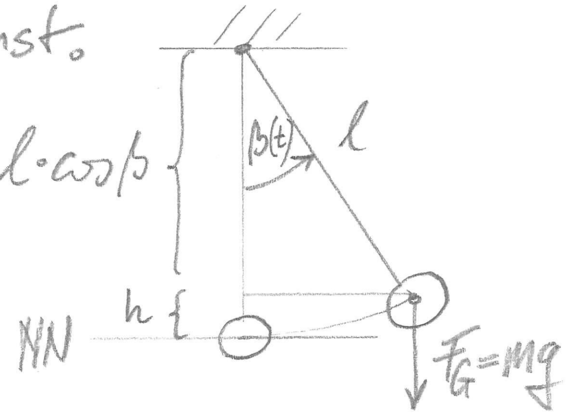
Herleitung einer Formel für die Periodendauer beim anharmonischen Fadenpendel

EES: $$\underbrace{\frac{mv^2}{2}}_{=E_K} + \underbrace{mg\cdot l\cdot(1-\cos\beta)}_{=E_{POT}} = \text{const.}$$



$$v = l\cdot\dot\beta\ ,\quad \omega^2 = \frac{g}{l}$$

$$\frac{ml^2}{2}\dot\beta^2 + mgl(1-\cos\beta) = \text{const.}$$

max. Auslenkg

$$\frac{\cancel{m}l^{\cancel{2}}}{2}\dot\beta^2 + \cancel{m}g\cancel{l}(1-\cos\beta) = \cancel{m}g\cancel{l}(1-\cos\beta_0)\ \Big|:l$$

$$\frac{1}{2}\dot\beta^2 + \frac{g}{l}(1-\cos\beta) = \frac{g}{l}(1-\cos\beta_0)$$

$$\frac{1}{2}\dot\beta^2 + \cancel{\omega^2} - \omega^2\cos\beta = \cancel{\omega^2} - \omega^2\cos\beta_0$$

$$\frac{1}{2}\dot\beta^2 = \omega^2(\cos\beta - \cos\beta_0)$$

$$\Longrightarrow \frac{d\beta}{dt} = (\pm)\sqrt{2\omega^2(\cos\beta-\cos\beta_0)}$$

$$\frac{d\beta}{\sqrt{2\omega^2(\cos\beta-\cos\beta_0)}} = dt \quad \Big| \int_0^{\beta_0}\ldots$$

$$\int_0^{\beta_0}\frac{d\beta}{\sqrt{2\omega^2(\cos\beta-\cos\beta_0)}} = \frac{T}{4}$$

$$\Longrightarrow T = 4\cdot\int_0^{\beta_0}\frac{d\beta}{\sqrt{2\omega^2(\cos\beta-\cos\beta_0)}}$$ … elliptisches Int. 1. Art

Taylor: $\cos\beta = 1 - \frac{\beta^2}{2} + \frac{\beta^4}{24} + O(t^6)$

$$\Rightarrow \cos\beta - \cos\beta_0 \approx \cancel{1} - \frac{\beta^2}{2} + \frac{\beta^4}{24} - \cancel{1} + \frac{\beta_0^2}{2} - \frac{\beta_0^4}{24} =$$

$$= \frac{\beta_0^2}{2} - \frac{\beta^2}{2} + \frac{\beta^4}{24} - \frac{\beta_0^4}{24} =$$

$$= \frac{\beta_0^2}{2} - \frac{\beta^2}{2} - \left(\frac{\beta_0^4}{24} - \frac{\beta^4}{24}\right) =$$

$$= \frac{1}{2}(\beta_0^2 - \beta^2) - \frac{1}{12} \cdot \frac{1}{2} \cdot (\beta_0^2 - \beta^2)(\beta_0^2 + \beta^2) =$$

$$= \frac{1}{2}(\beta_0^2 - \beta^2) \cdot \left(1 - \frac{1}{12}(\beta_0^2 + \beta^2)\right)$$

$$\Rightarrow T = 4 \cdot \int_0^{\beta_0} \frac{d\beta}{\sqrt{\cancel{2}\omega^2 \cdot \frac{1}{\cancel{2}}(\beta_0^2 - \beta^2) \cdot \left(1 - \frac{1}{12}(\beta_0^2 + \beta^2)\right)}} =$$

$$= \frac{4}{\omega} \int_0^{\beta_0} \frac{d\beta}{\sqrt{\beta_0^2 - \beta^2} \cdot \sqrt{1 - \frac{1}{12}(\beta_0^2 + \beta^2)}}$$

NW: Taylor: wenn $x \downarrow \Rightarrow \frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x$

mit $x = \frac{1}{12}(\beta_0^2 + \beta^2)$

$$\Rightarrow T = \frac{4}{\omega} \int_0^{\beta_0} \frac{1}{\sqrt{\beta_0^2 - \beta^2}} \cdot \left(1 + \frac{1}{24}(\beta_0^2 + \beta^2)\right) d\beta$$

$$T = \frac{4}{\omega} \underbrace{\int_0^{\beta_0} \frac{d\beta}{\sqrt{\beta_0^2 - \beta^2}}}_{I_1} + \frac{\overset{1}{\cancel{4}}}{\omega} \frac{1}{\underset{6}{\cancel{24}}} \underbrace{\int_0^{\beta_0} \frac{\beta_0^2 + \beta^2}{\sqrt{\beta_0^2 - \beta^2}} d\beta}_{I_2}$$

$$I_1 = \int_0^{\beta_0} \frac{\frac{1}{\beta_0}\, d\beta}{\frac{1}{\beta_0}\sqrt{\beta_0^2-\beta^2}} = \frac{1}{\beta_0}\int_0^{\beta_0} \frac{d\beta}{\sqrt{1-\frac{\beta^2}{\beta_0^2}}}$$

Subst: $x^2 = \frac{\beta^2}{\beta_0^2} \Rightarrow dx = \frac{1}{\beta_0}\, d\beta \Rightarrow d\beta = \beta_0\, dx$

$$I_1 = \frac{1}{\cancel{\beta_0}} \int_{\dots}^{\dots} \frac{\cancel{\beta_0}\, dx}{\sqrt{1-x^2}} = \arcsin x \Big|_{\dots}^{\dots} \underset{\text{Rücksubst}}{=} \arcsin \frac{\beta}{\beta_0}\Big|_0^{\beta_0} =$$

$$= \arcsin 1 - \arcsin 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$I_2 = \int_0^{\beta_0} \frac{\beta_0^2+\beta^2}{\sqrt{\beta_0^2-\beta^2}}\, d\beta$$

Subst: $\beta = \beta_0 \cdot \sin x \quad d\beta = \beta_0 \cdot \cos x \cdot dx$

$x = \arcsin\left(\frac{\beta}{\beta_0}\right)$

UG: $\beta = 0$

$x_{UG} = \arcsin(0) = 0$

OG: $\beta = \beta_0$

$x_{OG} = \arcsin\left(\frac{\beta_0}{\beta_0}\right) = \arcsin(1) = \frac{\pi}{2}$

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{\beta_0^2+\beta_0^2\sin^2 x}{\sqrt{\beta_0^2-\beta_0^2\sin^2 x}} \cdot \beta_0 \cdot \cos x\, dx =$$

$$= \int_0^{\frac{\pi}{2}} \frac{\beta_0^2+\beta_0^2\sin^2 x}{\cancel{\beta_0}\underbrace{\sqrt{1-\sin^2 x}}_{=\cancel{\cos x}}} \cancel{\beta_0}\,\cancel{\cos x}\, dx =$$

$$= \int_0^{\frac{\pi}{2}} \left(\beta_0^2 + \beta_0^2 \sin^2 x\right) dx = \beta_0^2 \int_0^{\frac{\pi}{2}} dx + \beta_0^2 \int_0^{\frac{\pi}{2}} \sin^2 x\, dx =$$

$$= \beta_0^2 \cdot \frac{\pi}{2} + \beta_0^2 \cdot \frac{\pi}{4} = \beta_0^2\left(\frac{2\pi}{4} + \frac{\pi}{4}\right) = \beta_0^2 \cdot \frac{3\pi}{4}$$

$$\Rightarrow T = \frac{\cancel{4}}{\omega} \cdot \frac{\pi}{\cancel{2}} + \frac{1}{\omega} \cdot \frac{1}{\cancel{6}_{2}} \cdot \beta_0^2\, \frac{\cancel{3}\pi}{4} = 2\pi \cdot \sqrt{\frac{l}{g}} + \frac{\pi}{8} \cdot \beta_0^2 \cdot \sqrt{\frac{l}{g}}$$

$$T = 2\pi \cdot \sqrt{\frac{l}{g}} \cdot \left(1 + \frac{1}{16}\beta_0^2 + \dots\right)$$