Von der Pendelkette (longitudinal) zur Boussinesq-Gleichung

Das Kraftgesetz (diskret & kontinuierlich)

$$F(l-l_0) = C\cdot\left((l-l_0)+\alpha(l-l_0)^2\right) \quad \dots \text{ diskret}$$

$$\sigma = E\cdot(\varepsilon+\alpha\varepsilon^2) = E\cdot\left(\frac{l-l_0}{l_0}+\alpha\left(\frac{l-l_0}{l_0}\right)^2\right)$$

$$\sigma = \frac{F}{A} \Rightarrow F = \sigma A = EA(\varepsilon+\alpha\varepsilon^2) \quad \dots \text{ kontinuierlich}$$

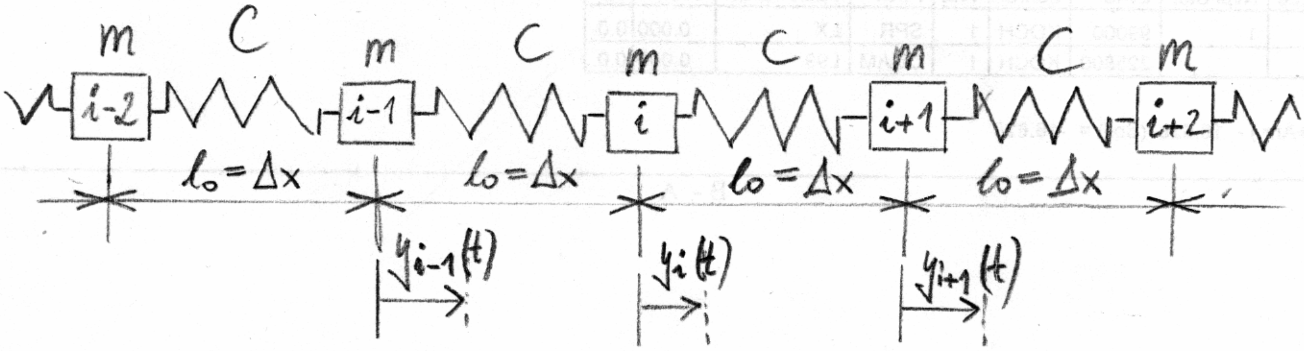
$$\rightarrow C\cdot\left((l-l_0)+\alpha(l-l_0)^2\right) = EA\left(\frac{l-l_0}{l_0}+\alpha\frac{(l-l_0)^2}{l_0^2}\right) \quad \Big| :(l-l_0)$$

$$C\cdot\left(1+\alpha(l-l_0)\right) = EA\left(\frac{1}{l_0}+\alpha\frac{l-l_0}{l_0^2}\right)$$

$$\Rightarrow C = EA\cdot\frac{\frac{1}{l_0}+\alpha\frac{l-l_0}{l_0^2}}{1+\alpha(l-l_0)} =$$

$$= EA\cdot\frac{\frac{l_0+\alpha(l-l_0)}{l_0^2}}{1+\alpha(l-l_0)} =$$

$$= EA\cdot\frac{l_0+\alpha(l-l_0)}{l_0^2\left(1+\alpha(l-l_0)\right)} \approx EA\cdot\frac{\cancel{l_0}}{l_0^{\cancel{2}}} = \frac{EA}{l_0}$$



Bewegungsglg:

$$m\cdot\ddot{y}_i = C\left((y_{i+1}-y_i)+\alpha(y_{i+1}-y_i)^2\right) - C\left((y_i-y_{i-1})+\alpha(y_i-y_{i-1})^2\right)$$

Übergang diskret $\to$ kontinuierlich

$$m = \rho \cdot A \cdot \Delta x \quad \ldots \quad \text{„Saitenelement"}$$

$$y_{i-1}(t) \to y(x-\Delta x,t),\; y_i(t) \to y(x,t),\; y_{i+1}(t) \to y(x+\Delta x,t)$$

$$\Rightarrow \rho \cdot A \cdot \Delta x\, \ddot{y}(x,t) = C\Big(y(x+\Delta x,t) - 2y(x,t) + y(x-\Delta x,t)\Big)$$
$$+ C \cdot \alpha\Big(\big(y(x+\Delta x,t) - y(x,t)\big)^2 - \big(y(x,t) - y(x-\Delta x,t)\big)^2\Big)$$

Taylor:

$$y(x+\Delta x,t) = y(x,t) + \Delta x \cdot y'(x,t) + \frac{\Delta x^2}{2} y''(x,t) + \frac{\Delta x^3}{6} y'''(x,t) + \frac{\Delta x^4}{24} y''''(x,t) + \mathcal{O}(\Delta x^5)$$

$$y(x-\Delta x,t) = y(x,t) - \Delta x \cdot y'(x,t) + \frac{\Delta x^2}{2} y''(x,t) - \frac{\Delta x^3}{6} y'''(x,t) + \frac{\Delta x^4}{24} y''''(x,t) + \mathcal{O}(\Delta x^5)$$

$$\Rightarrow \rho A \Delta x\, \ddot{y}(x,t) = \overbrace{\frac{EA}{\Delta x}}^{=C} \cdot \left(\Delta x^2 y''(x,t) + \frac{\Delta x^4}{12} y''''(x,t)\right)$$
$$+ \frac{EA}{\Delta x} \alpha \cdot \left(\left(\Delta x\, y'(x,t) + \frac{\Delta x^2}{2} y''(x,t) + \frac{\Delta x^3}{6} y'''(x,t) + \ldots\right)^2 - \left(\Delta x\, y'(x,t) - \frac{\Delta x^2}{2} y''(x,t) + \frac{\Delta x^3}{6} y'''(x,t) + \ldots\right)^2\right)$$

$$\rho A \Delta x\, \ddot{y}(x,t) = \frac{EA}{\Delta x}\left(\Delta x^2 \cdot y''(x,t) + \frac{\Delta x^4}{12} y''''(x,t)\right)$$
$$+ \frac{EA}{\Delta x} \alpha \Big(\underbrace{\Delta x^2 y'^2(x,t) - \Delta x^2 y'^2(x,t)}_{=0} + \underbrace{\cancel{2} \cdot \frac{\Delta x^3}{\cancel{2}} y'(x,t) \cdot y''(x,t) + \ldots + \cancel{2} \frac{\Delta x^3}{\cancel{2}} y'(x,t) y''(x,t) + \ldots}_{= 2 \Delta x^3 y'(x,t) \cdot y''(x,t) = \Delta x^3 \cdot \cancel{2} \cdot \frac{1}{\cancel{2}} \left(y'^2(x,t)\right)'} \Big)$$

mit $y'(x,t) \cdot y''(x,t) = \frac{1}{2}\left(y'^2(x,t)\right)' \quad \Rightarrow$

$$\rho \cancel{A} \Delta x \, \ddot{y}(x,t) = \frac{E\cancel{A}}{\Delta x} \cdot \left(\Delta x^2 y''(x,t) + \frac{\Delta x^4}{12} y''''(x,t) + \alpha \Delta x^3 \left(y'^2(x,t) \right)' \right)$$

$$\ddot{y}(x,t) = \frac{E}{\rho} \left(y''(x,t) + \frac{\Delta x^2}{12} y''''(x,t) + \alpha \Delta x \left(y'^2(x,t) \right)' \right)$$